Task 1 (80 points)
Consider the functions $F(t)$ and $G(t)$,

$$F(t) = \frac{1}{b} \exp\left(-\frac{(t-t_0)^2}{a^2}\right), \qquad G(t) = \frac{a}{1+b\,t^2}\,. \tag{1}$$

Assume that a and b are real.

(a.) Calculate $||F||$. Express b in terms of a so that F is normalized to unity.

(b.) Calculate $\langle t \rangle$ and $\langle t^2 \rangle$.

(c.) Calculate the Fourier transform $\widetilde{F}(\omega)$.

(d.) Calculate $\langle \omega \rangle$ and $\langle \omega^2 \rangle$.

(e.) Verify that the relation $\Delta t \cdot \Delta\omega \gtrsim \frac{1}{2}$ is fulfilled for the given example.

Now, repeat tasks (a.)—(e.) for the function $G(t)$. For certain integrals, you may use Wolfram Alpha.

Task 2 (30 points)
If $h(t) = f(t)\,g(t)$, show that for the Fourier transforms $\widetilde{f}$ and $\widetilde{g}$ fulfill

$$\widetilde{h}(\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega'}{2\pi}\, \widetilde{f}(\omega')\, \widetilde{g}(\omega-\omega')\,, \tag{2}$$

and if $\widetilde{h}(\omega) = \widetilde{f}(\omega)\,\widetilde{g}(\omega)$, then

$$h(t) = \int \mathrm{d}t'\, f(t')\, g(t-t')\,. \tag{3}$$

I.e., show that multiplication in coordinate space/time is equivalent to a convolution in Fourier space, and vice versa.

Task 3 (30 points)
Calculate the following functions:

$$\vec{F}_1(\vec{r}) = \vec{\nabla}\, C\, \exp(-a|\vec{r}-\vec{r}'|)\,, \tag{4}$$

$$\vec{F}_2(\vec{r}) = \vec{\nabla}\, \left[C\,|\vec{r}-\vec{r}'|\, \exp(-a|\vec{r}-\vec{r}'|)\right]\,, \tag{5}$$

$$\vec{F}_3(\vec{r}) = \vec{\nabla}^2\, \left[C\,|\vec{r}-\vec{r}'|\, \exp(-a|\vec{r}-\vec{r}'|)\right]\,, \tag{6}$$

Here, a and C are constants. We observe that $\vec{\nabla} = \partial/\partial\vec{r}$. Which physical dimension do these constants (a and C) have, if we postulate that F_1, F_2 and F_3 should be dimensionless?

The tasks are due Thursday, 03–OCT–2023.