Task 1 (50 points)
You are given the electric field and current

$$\vec{E}(\vec{r},t) = E_0 \cos(k(x-ct))\,\hat{e}_z\,, \tag{1}$$

$$\vec{J}(\vec{r},t) = \vec{0}\,. \tag{2}$$

The electric field describes a cw wave of an electromagnetic wave, z polarized, that travels in the positive x direction.
(a) Show that $\vec{\nabla}\cdot\vec{E}(\vec{r},t) = 0$. (b) Calculate $\rho(\vec{r},t)$. (c) Find $\vec{B}(\vec{r},t)$ from the relation

$$\vec{\nabla}\times\vec{E}(\vec{r},t) + \frac{\partial}{\partial t}\vec{B}(\vec{r},t) = 0\,. \tag{3}$$

(d) Using your (anticipated) result,

$$\vec{B}(\vec{r},t) = -\frac{E_0}{c}\cos(k(x-ct))\,\hat{e}_y\,, \tag{4}$$

show that

$$\vec{\nabla}\times\vec{B}(\vec{r},t) - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \vec{0}\,. \tag{5}$$

Now, repeat tasks (a)—(d) under the assumption $\vec{E}(\vec{r},t) = E_0 \cos(k_0(x-ct))\,\hat{e}_y$, i.e., for a y-polarized electromagnetic wave.

Task 2 (30 points)
You are given the same real rather than complex electric and magnetic fields as Eqs. (1) and (4) (with the cosine), and again $\vec{J}(\vec{r},t) = \vec{0}$. Recall the Poynting vector,

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t)\,. \tag{6}$$

Recall the energy density of the electromagnetic field ($\epsilon_r = \mu_r = 1$),

$$u(\vec{r},t) = \frac{1}{2}\left[\epsilon_0\,\vec{E}^{\,2}(\vec{r},t) + \frac{1}{\mu_0}\vec{B}^{\,2}(\vec{r},t)\right]\,, \tag{7}$$

the local power density (characterizing the mechanical power density for the work done on the sample),

$$\frac{\partial P(\vec{r},t)}{\partial V} = \vec{E}(\vec{r},t)\cdot\vec{J}(\vec{r},t)\,, \tag{8}$$

and Poynting's theorem,

$$\vec{\nabla}\cdot\vec{S}(\vec{r},t) = -\frac{\partial u(\vec{r},t)}{\partial t} - \frac{\partial P(\vec{r},t)}{\partial V}\,. \tag{9}$$

(a) Calculate

$$u(\vec{r},t)\,, \qquad \frac{\partial P(\vec{r},t)}{\partial V}\,, \qquad \vec{\nabla}\cdot\vec{S}(\vec{r},t)\,. \tag{10}$$

(b) Average your answer over one period of the laser oscillation, $T = 2\pi/\omega$, with $\omega = ck$, verify that Poynting's theorem is fulfilled, and verify that your answers are equivalent to those you would get from the complex formalism where you promote the fields to complex variables which represent Fourier components of positive frequency.

The tasks are due Tuesday, 03–OCT–2023.