Task 1 (30 points). Consider the signals

$$\vec{E}\,(\vec{r},t) = \vec{E}_0(\vec{r})\,\cos(\omega_0\,t)\,, \qquad \vec{J}\,(\vec{r},t) = \vec{J}_0(\vec{r})\,\cos(\omega_0\,t)\,, \tag{1}$$

with real rather than complex functions $\vec{E}_0(\vec{r})$ and $\vec{J}_0(\vec{r})$, and the angular frequency $\omega_0 = 2\pi/T$, where T is the oscillation period. Show that

$$\widetilde{\vec{E}}(\vec{r},\omega) = \pi\vec{E}_0(\vec{r})\,\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\,, \qquad \widetilde{\vec{J}}(\vec{r},\omega) = \pi\vec{J}_0(\vec{r})\,\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\,. \tag{2}$$

Task 2 (20 points). Consider the signals

$$\vec{E}\,(\vec{r},t) = \vec{E}_0(\vec{r})\,\exp(-\mathrm{i}\,\omega_0\,t)\,, \qquad \vec{J}\,(\vec{r},t) = \vec{J}_0(\vec{r})\,\exp(-\mathrm{i}\,\omega_0\,t)\,, \tag{3}$$

with real rather than complex functions $\vec{E}_0(\vec{r})$ and $\vec{J}_0(\vec{r})$, and the angular frequency $\omega_0 = 2\pi/T$, where T is the oscillation period. Repeat task #1, i.e., calculate the Fourier transforms. $\widetilde{\vec{E}}(\vec{r},\omega)$ and $\widetilde{\vec{J}}(\vec{r},\omega)$.

Task 3 (60 points). Investigate the following current density. For an oscillating current in an antenna located at $x = y = 0$ and confined to the interval $-a/2 < z < a/2$, we have

$$\vec{J}\,(\vec{r},t) = \vec{J}\,(x,y,z,t) = \mathrm{Re}\left\{\mathrm{e}^{-\mathrm{i}\,\omega\,t}\,i_0\,\left[\Theta\left(z+\frac{a}{2}\right)-\Theta\left(z-\frac{a}{2}\right)\right]\delta\,(x)\,\delta\,(y)\right\}\hat{\mathrm{e}}_z\,. \tag{4}$$

(a) Which physical dimension does i_0 have?
(b) For the given current density vector field, obtain the related charge density (charge conservation) by integrating its divergence with respect to time. (c) Obtain the longitudinal component of $\vec{J}$ by calculating its divergence and then calculating the integral

$$\vec{J}_{\parallel}\,(\vec{r},t) = -\frac{1}{4\pi}\vec{\nabla}\int\frac{\vec{\nabla}'\cdot\vec{J}\,(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r'\,. \tag{5}$$

Your answer will be valid for $(x,y,z) \neq (0,0,z')$ with $z' \in (-\frac{1}{2}a, \frac{1}{2}a)$. (d) Obtain the transverse component of $\vec{J}$ by calculating its curl and then calculating the integral

$$\vec{J}_{\perp}\,(\vec{r},t) = \frac{1}{4\pi}\vec{\nabla}\times\int\frac{\vec{\nabla}'\times\vec{J}\,(\vec{r}\,',t)}{|\vec{r}-\vec{r}\,'|}\,\mathrm{d}^3r' \tag{6}$$

explicitly, in Cartesian coordinates. Your answer will be valid for $(x,y,z) \neq (0,0,z')$ with $z' \in (-\frac{1}{2}a, \frac{1}{2}a)$.
(e) Verify that $\vec{J}(\vec{r},t) = \vec{J}_{\parallel}(\vec{r},t) + \vec{J}_{\perp}(\vec{r},t) = \vec{0}$ outside of the antenna, i.e., for $(x,y,z) \neq (0,0,z')$ with $z' \in (-\frac{1}{2}a, \frac{1}{2}a)$.

Task 4 (30 points). Calculate the Fourier transform of the Coulomb potential

$$V\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0\,|\vec{r}|} \tag{7}$$

with respect to both $\vec{r}$ and t, i.e., find $\widetilde{V}(\vec{k},\omega)$. Then, generalize you result to the Fourier transform of the following two potentials,

$$V(\vec{r},t;\vec{r}\,') = \frac{q}{4\pi\epsilon_0\,|\vec{r}-\vec{r}\,'|}\,, \qquad V\,(\vec{r},t) = \frac{q}{4\pi\epsilon_0\,|\vec{r}|}\,\exp(-\lambda\,|\vec{r}|)\,. \tag{8}$$

with respect to $\vec{r}$ and t. Which physical dimension must be associated with the parameter λ? **Hint:** Your answer for the first potential could look similar to $\widetilde{V}(\vec{k},\omega;\vec{r}\,') = 2\pi\delta(\omega)\,\frac{q}{\epsilon_0\,\vec{k}^2}\,\exp(\mathrm{i}\,\vec{k}\cdot\vec{r}\,')$. You may have to use so-called convergent factors. What is a convergent factor? Well, that's something for you to look up unless you know it already. Can you make some general statements about shifts in coordinate space and phases in Fourier space?

The tasks are due Tuesday, 19th of September 2023, with a possible extension to Thursday, 21st of September 2023.