Task 1 (20 points)
Consider a function of the form $W(\vec{r},t) = f(kz - \omega\, t)$, where z is the z coordinate, k is wave vector, t is the time, and ω is the angular frequency, satisfies the homogeneous form of the scalar wave equation. Show that *(i)*

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) W\,(\vec{r},t) = 0\,, \tag{1}$$

provided $\omega/k = c$. Show *(ii)* that the same property holds for a function $W(\vec{r},t) = f(k_x\, x + k_y\, y + k_z z - \omega\, t)$, provided that the parameters ω and k fulfill the relationship $\omega/\sqrt{k_x^2 + k_y^2 + k_z^2} = c$.

Here, f can be any function, e.g., $f(\xi) = a\,\cos(b\,\xi)$ with constant a and b, or $f(\xi) = c\,\exp(-d\,\xi^2)$ with constant c and d.

Task 2 (20 EXTRA points)
How would the function f in the above exercise have to be chosen in order to (a) represent an infinitely extended, plane wave, or (b) in order to represent a wave packet of finite size and finite duration (exponential decay for large time and/or distance). Choose a convenient functional form and verify your answer. Optional: plot your function at different times t and observe the motion of the wave packet. [This exercise is something to think about; more than one correct answer is possible!]

Task 3 (40 points)
You are given the space- and time-dependent vector potentials,

$$\Phi(\vec{r},t) = 0\,, \qquad \vec{A}(\vec{r},t) = \hat{\mathrm{e}}_z\,\frac{E_z}{\omega}\,\cos(\omega\, t - k_x\, x)\,. \tag{2}$$

(a) Calculate the electric and magnetic fields. (b) Verify that the Maxwell equations are fulfilled, and calculate $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$. (c) Visualize the solution and/or describe/interpret the physical properties of the fields.

Task 4 (30 points)
You are given the identity

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{J}\,(\vec{r},t)\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{J}\,(\vec{r},t)\right) - \vec{\nabla}^2\vec{J}\,(\vec{r},t)\ . \tag{3}$$

Derive this identity on the basis of the Levi-Cività tensor ϵ_{ijk} and the identity $\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$. The Einstein summation convention is being used.

Task 5 (30 points)
Show that a suitable vector potential for a static magnetic field $\vec{B}_0$ is $\vec{A}(\vec{r}) = \frac{1}{2}\left(\vec{B}_0\times\vec{r}\right)$, i.e., that $\vec{\nabla}\times\vec{A}(\vec{r}) = \vec{B}_0$. Show that, for $\vec{B}_0 = B_z\hat{\mathrm{e}}_z$, the result specializes to

$$\vec{A}(\vec{r}) = -\frac{1}{2}\,B_z\,y\,\hat{\mathrm{e}}_x + \frac{1}{2}\,B_z\,x\,\hat{\mathrm{e}}_y\,. \tag{4}$$

Show that

$$\vec{A}'(\vec{r}) = B_z\,x\,\hat{\mathrm{e}}_y \tag{5}$$

leads to the same $\vec{B}$ field as $\vec{A}$. Then, find the gauge transform function Λ that leads from $\vec{A}$ to $\vec{A}'$.

The most important homework, never announced but always due, is to read and understand the lecture notes, and the material covered in the lecture, so that questions on them can be answered at the beginning of each session.

The tasks are due Thursday, 14–SEP–2022, with a possible extension to Tuesday, 19–SEP–2022,.