Task 1 (60 points) Please do the task absolutely independently!
We investigate complex numbers by way of example. Calculate the following quantities z_1, z_2, z_3, $|z_1|$, $|z_2|$, $|z_3|$, and θ_1, θ_2, and θ_3,

$$z_1 = (4.3 + \mathrm{i}\,5.3)^2 = |z_1| \, \exp(\mathrm{i}\theta_1)\,, \tag{1}$$

$$z_2 = \sqrt{4.3 + \mathrm{i}\,5.3} = |z_2| \, \exp(\mathrm{i}\theta_2)\,, \tag{2}$$

$$z_3 = (2.3 + \mathrm{i}\,7.3)^{1/2} = |z_3| \, \exp(\mathrm{i}\theta_3)\,. \tag{3}$$

In the case of multivalued functions, give the result on the principal branch of the Riemann sheet. For the latter task, define the concept of the branch cut of the complex square root and choose a branch cut.

Task 2 (60 points) Please do the task absolutely independently!
You are given the (3×3)-matrix

$$\mathbb{M} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{4}$$

Determine the eigenvalues and eigenvectors of $\mathbb{M}$.

Task 3 (80 points) Please do the task absolutely independently!
Consider the surface integral over a vector field $\vec{F} = \vec{F}(\vec{r})$,

$$I = \int_{\partial V} \mathrm{d}\vec{A} \cdot \vec{F}(\vec{r})\,. \tag{5}$$

Answer the questions: If V is a volume (which one? an arbitrary one?), what does ∂V mean? How is the infinitesimal surface area element $\mathrm{d}\vec{A}$ defined, what is its direction, and what is its modulus $|\mathrm{d}\vec{A}|$?

Now, consider a parameterization $\vec{r} = \vec{r}(\theta, \varphi)$ of the surface. (Here, θ and φ are two variables parameterizing the surface. Why do we use two and not one variable, or three variables, to parameterize the surface?) Show that the surface integral can be written as

$$I = \int \mathrm{d}\theta \int \mathrm{d}\varphi \left(\frac{\partial \vec{r}(\theta, \varphi)}{\mathrm{d}\theta} \times \frac{\partial \vec{r}(\theta, \varphi)}{\mathrm{d}\varphi} \right) \cdot \vec{F}(\vec{r}(\theta, \varphi))\,. \tag{6}$$

How are the limits chosen for the θ and φ, in the general case? What is the relation of the limits to the shape of the surface, in general?

Now, consider the surface integral

$$J = \int_{\partial S_R} \mathrm{d}\vec{A} \cdot \vec{\nabla} \left(-\frac{1}{4\pi |\vec{r}|} \right)\,, \tag{7}$$

where S_R is the inner volume of the sphere of radius R, and ∂S_R is the surface of the sphere (i.e., the sphere itself). Find a suitable parameterization $\vec{r} = \vec{r}(\theta, \varphi)$ of the points on the surface of the sphere of radius R. Then, calculate (we talk about the same J)

$$J = \int \mathrm{d}\theta \int \mathrm{d}\varphi \left(\frac{\partial \vec{r}(\theta, \varphi)}{\mathrm{d}\theta} \times \frac{\partial \vec{r}(\theta, \varphi)}{\mathrm{d}\varphi} \right) \cdot \frac{\vec{r}(\theta, \varphi)}{4\pi |\vec{r}(\theta, \varphi)|^3}\,, \tag{8}$$

step-by-step, as a surface integral, using *precisely* the formalism indicated, and *precisely* the formulas indicated.

As a last step, explain the significance of your considerations for the Green function of the Poisson equation.

The most important homework, never announced but always due, is to read and understand the lecture notes, and the material covered in the lecture, so that questions on them can be answered at the beginning of each session. The tasks are due on Tuesday, 29–AUG–2022. No extension whatsoever will be given.