Task 1 (40 points) Reconsider the variational problem of the calculation of the potential in a *cylindrical* capacitor, with a more complex trial potential (with the same boundary conditions as in the lecture)

$$w(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \rho) = w(\rho) = \mathcal{A} + \mathcal{B}\,\rho + \mathcal{C}\,\rho^2 + \mathcal{D}\,\rho^3 \,, \qquad w(\rho = b) = 0 \,, \qquad w(\rho = c) = V_0 \,. \tag{1}$$

Use the parameters $b = 0.2\,\text{cm}$ and $c = 0.8\,\text{cm}$ and plot the variational solution you obtain. Convince yourself that you obtain an even better approximation to the analytic solution than in the lecture, where we had used a three-parameter variational *ansatz*.

Task 2 (40 points) Consider a functional

$$S[f(x)] = \int_V F(f(x))\, \mathrm{d}x' \,, \tag{2}$$

and the definition of the functional derivative as

$$\frac{\delta S}{\delta f(x)} = \lim_{\epsilon \to 0} \int_V \frac{F\left(f(x') + \epsilon\, \delta(x' - x)\right) - F\left(f(x')\right)}{\epsilon}\, \mathrm{d}x' \,. \tag{3}$$

It is understood here that the variation of the derivative of f is the derivative of the variation, i.e., in F also depends on $f'(x)$, then one has to consider $F(f(x') + \epsilon\, \delta(x' - x), f'(x') + \epsilon\, \delta'(x' - x)) - F\left(f(x'), f'(x')\right)$ in the numerator of Eq. (3).

(a) How would you generalize this definition for a functional of a function of a three-dimensional argument $\vec{r}$ as opposed to a one-dimensional argument x, i.e., for a functional of the form

$$S[\psi(\vec{r})] = \int_V F(\psi(\vec{r}))\, \mathrm{d}^3 r \,. \tag{4}$$

Hint: You may have to use a three-dimensional Dirac-δ function.

(b) Show, *based on your extended definition from task* **(a.)**, that for

$$S[\psi(\vec{r})] = \int_V \left[\vec{\nabla}\psi(\vec{r})\right]^2 \mathrm{d}^3 r \,, \tag{5}$$

one has

$$\frac{\delta S}{\delta \psi(\vec{r})} = -2\, \vec{\nabla}^2 \psi(\vec{r}) \,. \tag{6}$$

Task 3 (40 points) Consider the energy functional

$$E[\Phi] = \frac{\epsilon_0}{2} \int \mathrm{d}^3 r \left|\vec{\nabla}\Phi(\vec{r})\right|^2 \tag{7}$$

of an electrostatic potential $\Phi = \Phi(\vec{r})$. Show that [you may use the previous results from Task 1]

$$\frac{\delta E[\Phi]}{\delta \Phi(\vec{r})} = -\epsilon_0 \vec{\nabla}^2 \Phi(\vec{r}) \,. \tag{8}$$

Now calculate the second functional derivative. Show and comment every step in your derivation of the result that

$$\frac{\delta^2 E[\Phi]}{\delta \Phi(\vec{r}\,')\, \delta \Phi(\vec{r})} = -\epsilon_0\, \delta^{(3)}(\vec{r} - \vec{r}\,')\, \vec{\nabla}^2 \,. \tag{9}$$

In which sense can we say that the second functional derivative is positive, i.e., that "$\frac{\delta^2 E[\Phi]}{\delta \Phi(\vec{r}\,')\, \delta \Phi(\vec{r})} > 0$"? Think about the "positivity of a matrix" and a connection of this concept to the positivity of all of its eigenvalues.

The tasks are due Thursday, 04–MAY–2023.