Task 1 (60 points) Consider, as in the lecture, the Laplace equation in a rectangle,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \Phi\,(x,y) = 0\,. \tag{1}$$

We choose boundary conditions so that $\Phi\,(x,y)$ vanishes on three of the four boundaries,

$$\Phi\,(0,y) = \Phi\,(a,y) = \Phi\,(x,0) = 0\,, \qquad \Phi\,(x,b) = \Phi_0\,. \tag{2}$$

Describe, in your own words, with appropriate explanatory remarks, how to obtain the result

$$\Phi\,(x,y) = \frac{4\Phi_0}{\pi} \sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)\,\pi x/a\right]}{2n+1} \, \frac{\sinh\left[(2n+1)\,\pi y/a\right]}{\sinh\left[(2n+1)\,\pi b/a\right]}\,. \tag{3}$$

That is to say, give a complete rederivation of the result obtained in the lecture (and/or, lecture notes). Use your own words and write as much explanatory text as possible. Derive the results

$$\lim_{\epsilon\to 0^+} \Phi(0, b-\epsilon) = 0\,, \qquad \lim_{\epsilon\to 0^+} \Phi(\epsilon, b) = \Phi_0\,. \tag{4}$$

In doing so, show that the convergence in non-uniform near the point $x = 0$, and $y = b$. You may want to use the results

$$\sum_{n=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\,(2n+1)\,x}}{2n+1} = \operatorname{arctanh}(\exp(\mathrm{i}\,x))\,, \quad \operatorname{arctanh}(\exp(\pm\mathrm{i}\eta)) = \frac{\pm\mathrm{i}\pi}{4} + \frac{\ln(2)}{2} - \frac{\ln(\eta)}{2} + \mathcal{O}(\eta)\,. \tag{5}$$

In the first equation x is arbitary, while in the second equation, η is supposed to be small against unity. Thus, show that the limit $x \to 0$ of the function

$$G(x) = \sum_{n=0}^{\infty} \frac{\sin\left[(2n+1)\,\pi x/a\right]}{2n+1} \tag{6}$$

is approached non-uniformly. Show that $G(0) = 0$, but $\lim_{\eta\to 0^+} G(\eta) = \frac{\pi}{4}$.

Task 2 (80 points) Investigate the potential given in Eq. (6.90) of the lecture notes,

$$\Phi(x,y) = \frac{V_0}{\mathrm{i}\pi} \, \ln\left(\frac{\mathrm{e}^{\mathrm{i}\pi y/b} + \mathrm{e}^{\pi x/b}}{\mathrm{e}^{-\mathrm{i}\pi y/b} + \mathrm{e}^{\pi x/b}}\right)\,. \tag{7}$$

Show that, for real and positive $x > 0$, and $0 < y < b$, the result for $\Phi(x,y)$ is real rather than complex. Investigate, once more, Eq. (6.90) of the lecture notes, which is given above in Eq. (7). Show that

$$\lim_{\epsilon\to 0^+} \Phi(\epsilon, b) = 0\,, \qquad \lim_{\epsilon\to 0^+} \Phi(0, b-\epsilon) = V_0\,, \tag{8}$$

The task is due Thursday, 27–APR–2023.