Task 1 (50 points)
Consider the parameters

$$\vec{r} = \vec{r}_1 = 5.5\,\hat{e}_x + 3.3\hat{e}_y + 2.3\hat{e}_z\,, \qquad \vec{r}\,' = \vec{r}_2 = 5.1\,\hat{e}_x + 3.1\hat{e}_y + 2.2\hat{e}_z\,. \tag{1}$$

Define the terms

$$\begin{aligned} T_\ell &= \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\theta',\varphi') \\ &= \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\,\frac{r_2^\ell}{r_1^{\ell+1}}\,Y_{\ell m}(\theta_1,\varphi_1)\,Y^*_{\ell m}(\theta_2,\varphi_2)\,, \end{aligned} \tag{2}$$

where the second line is just a trivial specialization of the first, to the case $\vec{r} = \vec{r}_1$ and $\vec{r}\,' = \vec{r}_2$, and we anticipate that $r_2 = r_<$, and $r_1 = r_>$ (why?). Write a computer symbolic program which calculates, explicitly and numerically,

$$T_\ell\,, \qquad 0 \le \ell \le 20\,. \tag{3}$$

Show that the sum converges slowly, and that the result

$$\sum_{\ell=0}^{20} T_\ell \qquad \text{approximates only 77\% of the full result for} \qquad \frac{1}{|\vec{r} - \vec{r}\,'|}\,. \tag{4}$$

Then, calculate the first 201 terms $0 \le \ell \le 200$ and show that the full result for $1/|\vec{r} - \vec{r}\,'|$ is obtained to better than 99% agreement.

Task 2 (30 points)
With definitions as in the lecture, we had encountered the following dipole term of the multipole expansion of the potential, expressed in Cartesian coordinates,

$$\Phi(\vec{r})|_{\ell=1} = \frac{\vec{r}\cdot\vec{p}}{4\pi\epsilon_0\,r^3}\,. \tag{5}$$

By converting the multipole moments q_{1m}, expressed in spherical coordinates, into Cartesian coordinates, show that

$$\Phi(\vec{r})|_{\ell=1} = \frac{1}{\epsilon_0}\sum_{m=-1}^{1}\frac{q_{1m}}{3}\,\frac{Y_{1m}(\theta,\varphi)}{r^2}\,, \tag{6}$$

for the general case, where $p = p_x\,\hat{e}_x + p_y\,\hat{e}_y + p_z\,\hat{e}_z$.

Task 3 (30 EXTRA!!! points)
With definitions as in the lecture, we had encountered the quadrupole term in the multipole expansion of the electrostatic potential as follows,

$$\Phi(\vec{r})|_{\ell=2} = \sum_{ij}\frac{1}{4\pi\epsilon_0}\,\frac{Q_{ij}\,(3r_i\,r_j - \delta_{ij}\,r^2)}{2r^5}\,. \tag{7}$$

By converting the multipole moments q_{2m}, expressed in spherical coordinates, into Cartesian coordinates, show that

$$\Phi(\vec{r})|_{\ell=2} = \frac{1}{\epsilon_0}\sum_{m=-2}^{2}\frac{q_{2m}}{5}\,\frac{Y_{2m}(\theta,\varphi)}{r^2}\,. \tag{8}$$

The tasks are due Thursday, 13–APR–2023.