This sheet is a derivation of certain useful identities on Bessel functions.

It contains a number of missing steps which you could fill in, in the sense of an ungraded exercise.

The missing steps really are quite straightforward to fill in.

Step #1: Bessel's differential equation is

$$x^2\, y''(x) + x\, y'(x) + (x^2 - n^2)\, y(x) = 0\,. \tag{1}$$

The independent solutions are $J_n(x)$ and $Y_n(x)$. The solution regular at the origin is $J_n(x)$. Show that for an arbitrary scale parameter λ, the function $J_n(\lambda\, x)$ fulfills

$$x \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] + \left(\lambda^2 x^2 - n^2\right)\, J_n(\lambda\, x) = 0\,. \tag{2}$$

Now assume that λ and μ are two distinct zeros of the Bessel function $J_n(x)$, i.e.,

$$J_n(\lambda) = J_n(\mu) = 0\,, \qquad \lambda \neq \mu\,, \qquad J_n(\lambda\, x) = J_n(\mu\, x) = 0 \qquad \text{for } x = 1 \tag{3}$$

Multiplying Eq. (2) by $J_n(\mu\, x)/x$, show that the following equation holds,

$$J_n(\mu\, x)\, \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] + \frac{\lambda^2 x^2 - n^2}{x}\, J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{4}$$

Show (how?) that the following relation also holds,

$$J_n(\lambda\, x)\, \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \frac{\mu^2 x^2 - n^2}{x}\, J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{5}$$

Furthermore, manipulating Eqs. (4) and (5), show that (how?)

$$J_n(\mu\, x)\, \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] - J_n(\lambda\, x)\, \frac{\mathrm{d}}{\mathrm{d}x}\left[x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \left(\lambda^2 - \mu^2\right)\, x J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{6}$$

Now, work with Eq. (6) and show that it can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\mu\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\lambda\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \left(\lambda^2 - \mu^2\right)\, x J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,. \tag{7}$$

Finally, integrate over $x \in (0,1)$, use the fact that $J_n(\lambda) = J_n(\mu) = 0$, and show that

$$\int_0^1 \mathrm{d}x\, x\, J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,, \qquad \lambda \neq \mu\,, \qquad J_n(\lambda) = J_n(\mu) = 0\,, \tag{8}$$

In doing so, treat the case $n = 0$ separately; it requires special attention at the lower limit of integration because $J_n(0) = \delta_{n0}$ for $n \in \mathbb{N}_0$. However, you can use the known fact that $J_n'(0) = 0$. In general,

$$J_n(x) = \frac{x^n}{2^n\, n!} + \mathcal{O}(x^{n+2})\,, \qquad x \to 0\,. \tag{9}$$

Step #2: From the recursion of the Bessel function

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{10}$$

and the formula

$$\frac{\mathrm{d}}{\mathrm{d}x} J_n(x) = \frac{1}{2} \left(J_{n-1}(x) - J_{n+1}(x)\right) \tag{11}$$

conclude that (how?), at a zero of the Bessel function,

$$\frac{\mathrm{d}}{\mathrm{d}x} J_n(x) = -J_{n+1}(x) \qquad \text{for } J_n(x) = 0\,. \tag{12}$$

In the formula given in Eq. (7),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\mu\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\lambda\, x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(\lambda\, x)\, x \frac{\mathrm{d}}{\mathrm{d}x} J_n(\mu\, x)\right] + \left(\lambda^2 - \mu^2\right) x J_n(\mu\, x)\, J_n(\lambda\, x) = 0\,, \tag{13}$$

set $\mu = \lambda + \epsilon$ and integrate over $x \in (0,1)$ to obtain (how?)

$$\left[J_n((\lambda+\epsilon))\, \lambda J_n'(\lambda)\right] + \left(\lambda^2 - (\lambda+\epsilon)^2\right) \int_0^1 \mathrm{d}x\, x\; J_n((\lambda+\epsilon)\, x)\; J_n(\lambda\, x) = 0\,, \tag{14}$$

Expand to first order in ϵ to obtain (how?)

$$\epsilon\, \lambda \left[J_n'(\lambda)\right]^2 - 2\, \lambda\, \epsilon \int_0^1 \mathrm{d}x\, x\; \left[J_n(\lambda\, x)\right]^2 = 0\,, \tag{15}$$

and thus show that

$$\int_0^1 \mathrm{d}x\, x\; \left[J_n(\lambda\, x)\right]^2 = \frac{1}{2} \left[J_{n+1}(\lambda)\right]^2\,. \tag{16}$$

This completes the result (8) for the special case $\lambda = \mu$.

As a last step, perform the scale transformation

$$\lambda \to \tilde{\lambda}\, a, \qquad x \to \rho/a\,, \tag{17}$$

to obtain

$$\int_0^a \mathrm{d}\rho\, \rho\; \left[J_n(\tilde{\lambda}\, \rho)\right]^2 = \frac{1}{2}\, a^2 \left[J_{n+1}(\tilde{\lambda}\, a)\right]^2\,, \qquad \text{for } J_n(\tilde{\lambda}\, a) = 0\,. \tag{18}$$

Step #3: Start once more from Eq. (7), but with the replacements $\lambda \to a$, and $\mu \to b$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(b\,x)\,x\frac{\mathrm{d}}{\mathrm{d}x}J_n(a\,x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[J_n(a\,x)\,x\frac{\mathrm{d}}{\mathrm{d}x}J_n(b\,x)\right] + \left(a^2 - b^2\right) xJ_n(a\,x)\,J_n(b\,x) = 0\,. \tag{19}$$

Verify, by looking at your favorite literature reference, that

$$J_n(\rho) \sim \sqrt{\frac{2}{\pi\,\rho}}\,\sin\left(\rho - \frac{(n-1/2)\,\pi}{2}\right)\,, \qquad \rho \to \infty\,. \tag{20}$$

This implies that $\rho = \infty$ is a zero of the Bessel function. Integrating, thus, Eq. (19) within the interval $x \in (0, \infty)$, show that (how?)

$$\int_0^\infty \mathrm{d}x\,x\,J_n(a\,x)\,J_n(b\,x) = 0\,, \qquad a \neq b\,. \tag{21}$$

You may have to treat the case $n = 0$ separately and observe that the slope of $J_0(x)$ vanishes at $x = 0$.

Now treat the limit $a \to b$. The only region which can sizeably contribute to the integral in this limit is the one for very large x; otherwise only a very small displacement $a = b + \epsilon$ will lead to a vanishing integral. Write the asymptotics (20) as an exponential,

$$J_n(a\,x) \sim \frac{1}{2\,\mathrm{i}}\sqrt{\frac{2}{\pi\,a\,x}}\left\{\exp\left[\mathrm{i}\left(a\,x - \frac{(n-1/2)\,\pi}{2}\right)\right] - \exp\left[-\mathrm{i}\left(a\,x - \frac{(n-1/2)\,\pi}{2}\right)\right]\right\}\,, \qquad x \to \infty\,, \tag{22}$$

$$J_n(b\,x) \sim \frac{1}{2\,\mathrm{i}}\sqrt{\frac{2}{\pi\,b\,x}}\left\{\exp\left[\mathrm{i}\left(b\,x - \frac{(n-1/2)\,\pi}{2}\right)\right] - \exp\left[-\mathrm{i}\left(b\,x - \frac{(n-1/2)\,\pi}{2}\right)\right]\right\}\,, \qquad x \to \infty\,. \tag{23}$$

Show that (how?) the only relevant terms in the integrand $x\,J_n(a\,x)\,J_n(b\,x)$ in Eq. (21) are given by the following replacement,

$$\begin{aligned} x\,J_n(a\,x)\,J_n(b\,x) &\to x\left(\frac{1}{2\,\mathrm{i}}\right)^2\sqrt{\frac{2}{\pi\,a\,x}}\sqrt{\frac{2}{\pi\,b\,x}}\,\{-\exp\left(\mathrm{i}\,(a-b)\;x\right) - \exp\left(-\mathrm{i}\,(a-b)\;x\right)\} \\ &\to \frac{1}{2\pi\,\sqrt{a\,b}}\,\{\exp\left(\mathrm{i}\,(a-b)\;x\right) + \exp\left(-\mathrm{i}\,(a-b)\;x\right)\}\,. \end{aligned} \tag{24}$$

Symmetrize the integrand for $a \to b$ within the interval $x \in (-\infty, \infty)$ to show that (how?)

$$\begin{aligned} \int_0^\infty \mathrm{d}x\,x\,J_n(a\,x)\,J_n(b\,x) &= \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}x\,\frac{1}{2\pi\,\sqrt{a\,b}}\,\{\exp\left(\mathrm{i}\,(a-b)\;x\right) + \exp\left(-\mathrm{i}\,(a-b)\;x\right)\} \\ &= \frac{1}{2\pi\,\sqrt{a\,b}}\,2\pi\delta(a-b) = \frac{1}{\sqrt{a\,b}}\,\delta(a-b) = \frac{1}{a}\,\delta(a-b)\,, \qquad a \to b\,. \end{aligned} \tag{25}$$

Furthermore, for $a \neq b$, this relation actually reduces to Eq. (21) for $a \neq b$, so it is valid for all a and b.

Step #4: Start from the relation

$$\int_0^\infty \mathrm{d}x\, x\, J_n(a\,x)\, J_n(b\,x) = \frac{1}{\sqrt{a\,b}}\,\delta(a-b)\,. \tag{26}$$

Spherical Bessel functions are defined as

$$j_\ell(\rho) = \sqrt{\frac{\pi}{2\,\rho}}\, J_{\ell+1/2}(\rho)\,. \tag{27}$$

Show that you can rewrite Eq. (26) as follows (how do you have to identify the variables?),

$$\int_0^\infty \mathrm{d}x\, x^2 \left(\sqrt{\frac{\pi}{2\,k\,x}}\, J_{\ell+1/2}(k\,x)\right) \left(\sqrt{\frac{\pi}{2\,k'\,x}}\, J_{\ell+1/2}(k'\,x)\right) = \left(\sqrt{\frac{\pi}{2}}\right)^2 \frac{1}{k\,k'}\,\delta(k-k')\,. \tag{28}$$

Finally, show that (how?)

$$\int_0^\infty \mathrm{d}x\, x^2\, j_\ell(k\,x)\, j_\ell(k'\,x) = \frac{\pi}{2\,k\,k'}\,\delta(k-k')\,. \tag{29}$$

The task is voluntary and for Geeks but is highly recommended.

It may also be helpful under special personal circumstances, e.g., when you are depressed anyway, the above exercise will serve as a means of increasing your depression, until your spirits come up again when you find the solution.