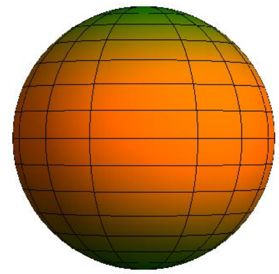
Task 1 (50 points)
Calculate the multipole decomposition and the electrostatic potential (monopole, dipole and quadrupole moments) for the charge distribution

$$\rho(\vec{r}) = \frac{Q}{a^2}\, \sin\theta\, \delta(r-a)\,. \tag{1}$$

Evaluate all multipoles of the given charge distribution with angular momenta $\ell = 0, 1, 2$ and write down the final potential. If red areas mark positive charge, blue areas denote negative charge, and neutral areas are green, then the sphere looks like the one on the right. Is this a sum of a monopole, and a quadrupole?

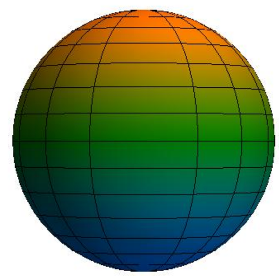


Hint: Maybe, we have $\Phi(\vec{r}) = \frac{\pi\, Q}{4\,\epsilon_0\, r} - \frac{\pi\, Q\, a^2}{64\,\epsilon_0\, r^3}\,\big(3\,\cos^2\theta - 1\big)$.

Task 2 (50 points)
Calculate the multipole decomposition and the electrostatic potential for the charge distribution

$$\rho(\vec{r}) = \frac{Q}{a^2}\, \cos\theta\, \delta(r-a)\,, \tag{2}$$

with a cosine instead of a sine, as compared to task 1. Is this a dipole charge distribution? For 50 extra points, calculate the electric field in spherical coordinates.



Hint: Maybe, we have $\Phi(\vec{r}) = \frac{Q\, a}{3\,\epsilon_0\, r^2}\, \cos\theta$.

Task 3 (50 points)
Calculate the electrostatic potential outside of the charge distribution

$$\rho(\vec{r}) = \frac{Q}{a^2}\, \left(3\cos^2\theta - \tfrac{1}{2}\right)\, \delta(r-a)\,. \tag{3}$$

Here, a is a parameter of dimension length (unit: meter), and Q is a parameter of dimension charge (unit: Coulomb). Show that $\rho(\vec{r})$ has the correct physical dimension of charge per volume, as required for a charge distribution. Conider the explicit form of a Y_{20} function, $Y_{20}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,\left(3\cos^2\theta - 1\right)$, but be aware of the term $-\frac{1}{2}$ in the charge distribution.

Task 4 (50 points)
One more charge distribution:

$$\rho(\vec{r}) = \frac{Q}{a^2}\, \sin^2\theta\, \delta(r-a)\,. \tag{4}$$

Please calculate the potential. You should obtain a monopole and a quadrupole term!

Hint: Maybe, we have $\Phi(\vec{r}) = \frac{2\,Q}{3\,\epsilon_0\, r} - \frac{Q\, a^2}{15\,\epsilon_0\, r^3}\,\big(3\,\cos^2\theta - 1\big)$.

Show all intermediate steps of your calculations.

The tasks are due Tuesday, 04–APR–2023.