Task 1 (40 points)
Verify the relation

$$\int Y^*_{\ell'\,m'}(\theta,\phi)\;Y_{\ell m}(\theta,\phi)\;\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi} Y^*_{\ell'\,m'}(\theta,\phi)\;Y_{\ell m}(\theta,\phi)\;\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \delta_{\ell\ell'}\,\delta_{mm'}\,, \tag{1}$$

by an explicit calculation for all spherical harmonics with $\ell = 0, 1$.

Task 2 (100 points)
In the lecture, we had calculated the potential $\Phi(\vec r)$ which solves the Poisson equation $\vec\nabla^2\Phi(\vec r) = (-q/\epsilon_0)\,\delta^{(3)}(\vec r - \vec r\,')$, and rewritten the result in terms of the Green function. Now, please repeat the derivation in the lecture, but in a different normalization. Namely, show that

$$g(\vec r - \vec r\,') = -\frac{1}{4\pi\,|\vec r - \vec r\,'|} = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\,\frac{r_<^{\ell}}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\;Y^*_{\ell m}(\theta',\varphi')\;, \tag{2}$$

by using (as in the lecture) the orthogonality of the spherical harmonics and the concatenation of the radial component of the Green function at $r = r'$.

Task 3 (100 points)
Find functions $\mathcal{A}(r')$ and $\mathcal{B}(r')$ so that

$$g_\ell(r - r') = -\frac{1}{2\ell+1}\,\frac{r_<^{\ell}}{r_>^{\ell+1}} = \mathcal{A}(r')\,r^{\ell} + \mathcal{B}(r')\,r^{-\ell-1}\,. \tag{3}$$

Hint: Consider the Heaviside Θ step function.

Task 4 (100 points)
Consider the parameters

$$\vec r = \vec r_1 = 4\,\hat{\mathrm{e}}_x + 2\hat{\mathrm{e}}_y + 7\hat{\mathrm{e}}_z\,, \qquad \vec r\,' = \vec r_2 = 0.1\,\hat{\mathrm{e}}_x + 0.2\hat{\mathrm{e}}_y + 0.4\hat{\mathrm{e}}_z\,. \tag{4}$$

Define the terms

$$\begin{aligned} T_\ell &= \sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r_<^{\ell}}{r_>^{\ell+1}}\,Y_{\ell m}(\theta,\varphi)\,Y^*_{\ell m}(\theta',\varphi') \\ &= \sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\,\frac{r_2^{\ell}}{r_1^{\ell+1}}\,Y_{\ell m}(\theta_1,\varphi_1)\,Y^*_{\ell m}(\theta_2,\varphi_2)\,, \end{aligned} \tag{5}$$

where the second line is just a trivial specialization of the first, to the case $\vec r = \vec r_1$ and $\vec r\,' = \vec r_2$, and we anticipate that $r_2 = r_<$, and $r_1 = r_>$ (why?). Write a computer symbolic program which calculates, explicitly and numerically,

$$T_{\ell=0,1,2,3,4,5,6,7,8}\,, \qquad \text{(nine terms)} \tag{6}$$

Show that the sum converges fast, and that the result

$$\frac{1}{|\vec r - \vec r\,'|} \approx \sum_{\ell=0}^{8} T_\ell \tag{7}$$

holds to at least six decimal figures.

The tasks are due Tuesday, 04–APR–2023.