Task 1 (50 points)
(a.) Define the concept of a Laurent series as opposed to a Taylor series. Define the concept of an Mth-order pole of a function of a complex variable, at a point $z = z_0$ in the complex plane. Show that the residue at an Mth-order pole can be calculated as follows,

$$\hat{\mathrm{Res}}_{z=z_0} f(z) = \frac{1}{(M-1)!} \left[\frac{\mathrm{d}^{M-1} \left\{ (z-z_0)^M \; f(z) \right\}}{\mathrm{d}z^{M-1}} \right]_{z=z_0} . \tag{1}$$

(b.) Additional task: Show that the above formula is consistent with what you expect for a first-order pole. In particular, **based on Eq.** (1), calculate the residues $[M = 1]$

$$R_1 = \hat{\mathrm{Res}}_{z=0} \frac{1}{\sin(z)} , \qquad R_2 = \hat{\mathrm{Res}}_{z=\pi} \frac{1}{\sin(z)} , \tag{2}$$

Show every step of your considerations accurately, and include all intermediate considerations.

Hint: Consult written distributed lecture notes.

Task 2 (40 points)
Calculate the following residues,

$$T_1 = \hat{\mathrm{Res}}_{z=0} \left(\frac{1}{z^3} \exp(-z^3) \sin(z) \right) , \qquad T_2 = \hat{\mathrm{Res}}_{z=0} \left(\frac{1}{z^3} \exp(-z^2) \right) , \tag{3}$$

$$T_3 = \hat{\mathrm{Res}}_{z=0} \left(\frac{1}{z^4} \exp(z) \cos(z) \right) , \qquad T_4 = \hat{\mathrm{Res}}_{z=0} \left(\frac{1}{[\sin(z)]^2} \right) . \tag{4}$$

Task 3 (40 points)
(a.) Calculate the integral

$$I = \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^2 + 36} = \frac{\pi}{6} \tag{5}$$

by residue calculus, mapping the integral onto a closed contour in the complex plane, and closing the contour, either in the upper or lower half of the complex plane. Any other way of analytic integration will result in zero points. **(b.)** Calculate the same integral by numerical integration, using your favorite computer system, and compare the result to $\pi/6 \approx 0.523599\ldots$.

Task 4 (40 points)
Consider the three-dimensional generalization of a Taylor expansion,

$$f(\vec{r}) = f(\vec{0}) + \vec{r} \cdot \vec{\nabla} f(\vec{r}) \Big|_{\vec{r}=\vec{0}} + \sum_{i,j=1}^{3} \frac{1}{2!} r_i \, r_j \; \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} f(\vec{r}) \Big|_{\vec{r}=\vec{0}} + \ldots ,$$

$$\vec{r} \cdot \vec{\nabla} f(\vec{r}) \Big|_{\vec{r}=\vec{0}} = \sum_{i=1}^{3} r_i \; \frac{\partial}{\partial r_i} f(\vec{r}) \Big|_{\vec{r}=\vec{0}} . \tag{6}$$

Consider the function

$$f(\vec{r}) = f(x, y, z) = \exp\left[-(x + x^2 + y^2 + 2z^2) \right] \tag{7}$$

at the point $x = 0.1$, $y = 0.1$, and $z = 0.05$. Evaluate the first two terms of the three-dimensional Tyalor expansion at the origin explicitly and show that this expansion leads to the approximation

$$f(0.1, 0.1, 0.05) = 0.882496 \cdots \approx 1 - 0.1 - 0.02 = 0.88 . \tag{8}$$

where the terms 0.1 and 0.02 are obtained from the two subleading terms in the Taylor expansion.

The tasks are due Tuesday, 14-FEB-2023. Have fun doing the problems!