

Task 1 (50 points)

Consider Stokes's theorem in three dimension, formulated for a general three-dimensional vector field $\vec{F} = \vec{F}(\vec{r})$, and an infinitesimal line element $d\vec{s}$, and area element $d\vec{A}$,

$$\oint_{\partial A} \vec{F}(\vec{s}) \cdot d\vec{s} = \oint_{\partial A} \vec{F}(\vec{s}(t)) \cdot \frac{d\vec{s}(t)}{dt} dt = \int_A (\vec{\nabla} \times \vec{F}(\vec{r})) \cdot d\vec{A}, \quad (1)$$

where $\vec{s} = \vec{s}(t)$ parameterizes the boundary of the surface and

$$F(\vec{r}) = F_x(\vec{r}) \hat{e}_x + F_y(\vec{r}) \hat{e}_y + F_z(\vec{r}) \hat{e}_z, \quad (2)$$

$$d\vec{s} = \hat{e}_x ds_x + \hat{e}_y ds_y + \hat{e}_z ds_z, \quad (3)$$

$$d\vec{A} = \hat{n} |d\vec{A}|, \quad (4)$$

and \hat{n} is the surface normal. In the lecture notes, the theorem is shown for the case of a z oriented surface normal, with reference to a small, but not infinitesimally small surface δA , i.e., when $\delta\vec{A} = \hat{n} \delta A$ is

- Case I: a rectangle, $\hat{n} = \hat{e}_z$, with edges at x and $x + \delta x$, as well as y and $y + \delta y$ (Lecture, done, nothing to do).

One can then applied the “patchwork principle” to conclude that the theorem also holds for a large oriented surface A with boundary ∂A . **Include a small write-up of the corresponding lecture notes.** Now, show **by an explicitly calculation** that the theorem also holds when $\delta\vec{A} = \hat{n} \delta A$ is

- Case II: a rectangle, $\hat{n} = \hat{e}_x$, with edges at y and $y + \delta y$, as well as z and $z + \delta z$,
- Case III: a rectangle, $\hat{n} = \hat{e}_y$, with edges at x and $x + \delta x$, as well as z and $z + \delta z$,

Then, explain how the considerations change for

- Case IV: a rectangle, $\hat{n} = -\hat{e}_x$, with edges at y and $y + \delta y$, as well as z and $z + \delta z$,
- Case V: a rectangle, $\hat{n} = -\hat{e}_y$, with edges at x and $x + \delta x$, as well as z and $z + \delta z$,

Hint: How do you have to orient the line integral over $d\vec{s}$ in each case?

Task 2 (30 points)

Complete the proof of Gauss's theorem for a general vector field $\vec{F} = \vec{F}(\vec{r})$,

$$\int_{\partial V} \vec{F}(\vec{r}) \cdot d\vec{A} = \int_V \vec{\nabla} \cdot \vec{F}(\vec{r}) dV, \quad (5)$$

with reference to a small, but not infinitesimally small cubic reference volume δV , with edges at x and $x + \delta x$, y and $y + \delta y$, and z and $z + \delta z$. **Consider all sides of the cubic reference volume δV and explain how the “Lego principle” completes the proof for a “macroscopic” reference volume V .**

Task 3 (20 points)

Explain how the application of Stokes's and Gauss's theorem transform the Maxwell equations from integral to differential form.

The tasks are due Tuesday, 07-FEB-2023. Have fun doing the problems!