Task 1 (30 points)
Show by explicit differentiation that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln\left(\frac{\sqrt{x^2+y^2}}{a}\right) = 0\,, \tag{1}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \frac{1}{\sqrt{x^2+y^2+z^2}} = 0\,, \tag{2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2}\right) \frac{1}{x^2+y^2+z^2+\xi^2} = 0\,, \tag{3}$$

provided $x \neq 0$, $y \neq 0$, $z \neq 0$ and $\xi \neq 0$. How is the derived result related to the Green functions of the Poisson equation in two, three and four dimensions? (Hint: For the first and the second of the above three equations, you may use your lecture notes and an appropriate notation of your choice, e.g., $r \equiv ||\vec{r}|| = \sqrt{x^2+y^2+z^2}$.)

Task 2 (30 points)
With the use of Gauss's theorem (divergence theorem), determine the prefactors which lead to solutions of the Poisson equations in two, three and four dimensions,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) g(x,y) = \delta^{(2)}(x,y)\,, \tag{4}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) g(x,y,z) = \delta^{(3)}(x,y,z)\,, \tag{5}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \xi^2}\right) g(x,y,z,\xi) = \delta^{(4)}(x,y,z,\xi)\,. \tag{6}$$

(Hint: You should formulate the divergence theorem in such a way that it is amenable to a generalization to four dimension. How would you paramerize a unit sphere imbedded in three dimensions? How would you paramerize a unit sphere imbedded in four dimensions?)

Task 3 (30 points)
Consider the function

$$f(x,y) = x^2 + y^2 + 3xy\,. \tag{7}$$

Show that its two-dimensional Taylor expansion, taken to second order, is equal to the function that you started from.

Task 4 (30 extra points)
Repeat **Task 3** for the function

$$f(x,y,z) = x + y + z + xy + x^3 + x^2 y + xyz\,, \tag{8}$$

in three-dimensional space. Show that it is reproduced by its third-order Taylor expansion.

Task 4 (30 points)
Calculate

$$Q_1 = \vec{\nabla}^2 \exp(-r/a)\,, \qquad Q_2 = \vec{\nabla} \exp(-r/a)\,, \qquad Q_2 = \vec{\nabla} \times [\hat{\mathrm{e}}_z \exp(-r/a)]\,, \tag{9}$$

where $r = \sqrt{x^2+y^2+z^2}$ and a is a constant of dimension length.

The tasks are due Tuesday, 07–FEB-2023. Have fun doing the problems!