Task 1 (45 points). Calculate the contour integrals

$$J_i = \oint_{C_i} \mathrm{d}z \, \frac{1}{z} \tag{1}$$

where $z \in \mathbb{C}$ is a complex variable, and the contours C_i with $i = 1, 2, 3$ are as follows:

- C_1 is a closed circular contour encircling the origin in the mathematically positive sense, with radius $R = |z| = 5$.
- C_2 is a closed circular contour encircling the origin in the mathematically negative sense, with radius $R = |z| = 3$.
- C_3 is a closed circular contour encircling the origin in the mathematically negative sense, with radius $R = |z| = 11$.

Give a reason why the following parameterizations could be considered:

$$C_1 = \{z = z(t) | z(t) = 5\, \exp(\mathrm{i}\, t),\ 0 < t < 2\pi\}\,, \tag{2a}$$
$$C_2 = \{z = z(t) | z(t) = 7\, \exp(-\mathrm{i}\, t),\ 0 < t < 2\pi\}\,, \tag{2b}$$
$$C_3 = \{z = z(t) | z(t) = 11\, \exp(-\mathrm{i}\, t),\ 0 < t < 2\pi\}\,. \tag{2c}$$

Calculate the contour integrals using an explicit parameterization $z = z(t)$, where t is the "time variable" along the contour. Any other solution will result in ZERO points. Also, show EVERY INTERMEDIATE STEP. This is very important.

Task 2 (55 points). **(a)** Let $f = f(z)$ be a complex function of a complex variable z, and let us assume that it can be written as $f = f_1 + \mathrm{i} f_2$, where f_1 and f_2 are the real and imaginary parts, respectively. Furthermore, let $\vec{F}^* = f_1\, \hat{\mathrm{e}}_x - f_2\, \hat{\mathrm{e}}_y$ be the "complex conjugate vector field". Show that

$$\begin{aligned} \oint f(z) \mathrm{d}z &= \oint (f_1 + \mathrm{i} f_2)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) \\ &= \int_{\partial A} \vec{F}^* \cdot \mathrm{d}\vec{\ell} + \mathrm{i} \int_A \vec{F}^* \cdot \mathrm{d}\vec{\ell}_\perp \\ &= \int_A (\vec{\nabla} \times \vec{F}^*)_z \mathrm{d}A + \mathrm{i} \int_A \vec{\nabla} \cdot \vec{F}^* \mathrm{d}A\,, \qquad \mathrm{d}\vec{\ell}_\perp = \mathrm{d}\vec{\ell} \times \hat{e}_z\,. \end{aligned} \tag{3}$$

All symbols are defined as in the lecture. **(b)** Find the vector field $\vec{F}^*$ corresponding to the function $f(z) = 1/z$ and find the path integrals over $\vec{F}^*$ which correspond to the contour integrals J_i over $f(z)$ with $i = 1, 2, 3$. You may consult your lecture notes. Yet, please use your own words in the derivation.

Task 3 (40 points). Show that the Green function of the two-dimensional Poisson equation is

$$G(x, y) = \frac{1}{2\pi} \ln\left(\frac{\sqrt{x^2 + y^2}}{a}\right) \qquad \vec{\nabla}^2 G(x, y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(x, y) = \delta^{(2)}(\vec{r})\,. \tag{4}$$

If possible, relate your result to Gauss's and Stokes's theorem, and Cauchy's theorem, in two dimensions, as done in the lecture.

The tasks are due Tuesday, 31–JAN–2023.