Task 1 (30 points)
Collect all notes you have from previous mathematical preparation courses, on Taylor and Laurent expansions, as well as differential equations and special functions (notably, Bessel functions). Try to refesh your memory on all aspects you may have forgotten in the meantime.

In particular, write a short essay on the question: How would you evaluate a line integral

$$I = \int_P \vec{F}(\vec{r}) \cdot \mathrm{d}\vec{r} \tag{1}$$

where P is a nontrivial path (say, with curves) and $\vec{F}(\vec{r})$ is a vector-valued function of a vector-valued variable $\vec{r}$ (the position). Hint: Start from a parameterisation $\vec{r} = \vec{r}_P(t)$, where t is the time and $\vec{r}_P(t)$ is the position of the object on the path P at time t.

How would your task become easier if

$$\vec{F}(\vec{r}) = -\vec{\nabla} f(\vec{r})\,, \tag{2}$$

where $f(\vec{r})$ is a scalar function (a potential). (Please note: The minus sign is inserted only for convention and has no special meaning.)

Task 2 (50 points)
Derive the Taylor expansion

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \mathcal{O}(x^4)\,. \tag{3}$$

Explain the meaning of the term $\mathcal{O}(x^4)$. Explain the connection of the Taylor expansion to the Einstein formula

$$E = \sqrt{\vec{p}^2\, c^2 + m^2 c^4} \tag{4}$$

with symbols explained as in the lecture. Where does the nonrelativistic kinetic energy $\vec{p}^2/(2m)$ appear in the formulas? Make a plot, using a computer algebra system of your choice, of the approximations

$$\sqrt{1+x} \approx 1 + \frac{x}{2}\,, \tag{5}$$

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}\,, \tag{6}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}\,, \tag{7}$$

and show, visually, how the approximations become better with increasing order.

The tasks are due Tuesday, 24-JAN-2023. Have fun doing the problems!